On Grzegorczyk's and Whitehead's definitions of point

Rafał Gruszczyński
Chair of Logic
Nicolaus Copernicus University in Toruń
gruszka@umk.pl

1 Introduction

One of the main tasks of philosophically oriented point-free theories of space is to deliver a plausible definition of *points*, entities which are assumed as primitives in point-based geometries and topologies. It is understood that—unlike in case of algebraically oriented theories of frames and locales—such definitions should be intuitive from the geometrical point of view, and should refer to objects which can be interpreted in the real world. Thus points are replaced by *regions* and the former are explained as abstractions from the latter. The main purpose of the talk is to draw a comparison between two seminal definitions of points: one of Andrzej Grzegorczyk's from [**?**] and the other of Alfred Whitehead's from [**?**].

2 Boolean contact algebras

About regions we assume that they form a Boolean algebra:

$$\mathfrak{R} = \langle R, \sqcap, \sqcup, -, \mathsf{0}, \mathsf{1} \rangle$$

which is turned into a Boolean *contact* algebra (BCA) by extending $\mathfrak{R}$ with a binary relation C of *contact* between regions, which satisfies the following axioms ($\not\mathsf{C}$ is the complement of C and $\leq$ is the standard ordering relation):

$$\mathsf{0} \not\mathsf{C}\, x\,, \tag{C0}$$

$$x \leq y \wedge x \neq \mathsf{0} \longrightarrow x \mathrel{\mathsf{C}} y\,, \tag{C1}$$

$$x \mathrel{\mathsf{C}} y \longrightarrow y \mathrel{\mathsf{C}} x\,, \tag{C2}$$

$$x \leq y \longrightarrow \forall_{z \in R}(z \mathrel{\mathsf{C}} x \longrightarrow z \mathrel{\mathsf{C}} y)\,, \tag{C3}$$

$$x \mathrel{\mathsf{C}} y \sqcup z \longrightarrow x \mathrel{\mathsf{C}} y \vee x \mathrel{\mathsf{C}} z\,. \tag{C4}$$

The class of all Boolean contact algebras will be denoted by '$\mathbf{BCA}$'. If C satisfies (**??**)–(**??**), it will be called *pre-contact* relation and the structure will bear the name of Boolean *pre-contact* algebra (BPCA). The class of all Boolean pre-contact algebras will be denoted by '$\mathbf{BPCA}$'.

In $\mathfrak{B} \in \mathbf{BPCA}$ we define an auxiliary relation of *non-tangential* inclusion (or *way below*, *well-inside*) relation:

$$x \ll y \;:\longleftrightarrow\; x \not\mathsf{C} -y\,. \tag{df $\ll$}$$

We accept the convention according to which given a class $\mathbf{K}$ of structures and some conditions $\varphi_1, \ldots, \varphi_n$ expressed in the language of $\mathbf{K}$, $\mathbf{K} + \varphi_1 + \ldots + \varphi_n$ (or $\mathbf{K} + \{\varphi_1, \ldots, \varphi_n\}$) is the subclass of $\mathbf{K}$ in which every structure satifies all $\varphi_1, \ldots, \varphi_n$, e.g.:

$$\mathbf{BCA} = \mathbf{BPCA} + (\textbf{??})\,.$$

We also define $x \circ y$ to mean that $x \sqcap y \neq \mathsf{0}$, and take $\perp \subseteq R \times R$ to be the set-theoretical complement of $\circ$. In the former case we say that x *overlaps* y, in the latter that x *is disjoint from* y or x *is incompatible with* y.

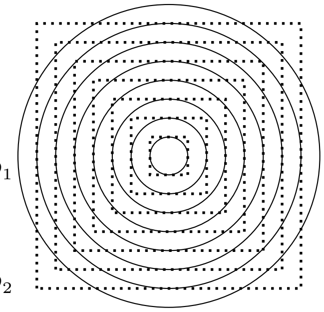

Figure 1: Q_1 and Q_2 representing the same location in space

3 Grzegorczyk points

A *Grzegorczyk representative of a point* (*G-representative*)[1] in $\mathfrak{B} \in \mathsf{BPCA}$ is a non-empty set Q of regions such that:

$$\mathsf{0} \notin Q\,, \tag{r0}$$

$$\forall_{u,v\in Q}(u = v \vee u \ll v \vee v \ll u)\,, \tag{r1}$$

$$\forall_{u\in Q}\exists_{v\in Q}\ v \ll u\,, \tag{r2}$$

$$\forall_{x,y\in R}\big(\forall_{u\in Q}(u \bigcirc x \wedge u \bigcirc y) \longrightarrow x \mathrel{\mathsf{C}} y\big)\,. \tag{r3}$$

Let $\mathbf{Q}_G$ be the set of all G-representatives of $\mathfrak{B}$. The purpose of the definition is to formally grasp the intuition of point as the system of diminishing regions determining a unique location in space. We call it a *representative*, since if we understand a point as perfect representation of some location in space, then it may happen that two different sets of regions represent one and the same location (see Figure **??** for geometrical intuitions on the Cartesian plane). We identify such G-representatives to represent the same location in space by taking as points filters generated by G-representatives or equivalence classes of G-representatives with respect to (defined further) co-initiality relation on sets of regions.

4 Whitehead points

A set of regions A is an *abstractive set* iff A satisfies (**??**), (**??**) and:

$$\neg\exists_{x\in A}\forall_{y\in A}\, x \leq y\,. \tag{A}$$

The class of all abstractive sets of a given BPCA will be denoted by '$\mathbf{A}$'. Since every abstractive set is a chain w.r.t. $\leq$, it must be the case that for every $x \in A$ there is $y \in A$ such that $y < x$. So, by the Axiom of Choice, every abstractive set is infinite.

The idea behind this definition is that we can abstract geometrical objects out of other entities. However, these entities do not have to be points, as in case of representatives of Grzegorczyk's, but might be planes, straight lines, triangles and so on. To use a simple example, we take the algebra $\mathrm{RO}(\mathbb{R})$ and the set of open intervals of the form:

$$\left\{\left(-\frac{n+1}{n}, \frac{n+1}{n}\right) \middle|\, n \in \omega \setminus \{0\}\right\}$$

which is an abstractive set and represents the closed interval $[-1, 1]$. Of course, we easily see that it is not a G-representative of a point, since regions $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{3}{2}\right)$ overlap all regions from the abstractive set, but are not in contact. So the set violates (**??**). In general, there are Grzegorczyk representatives which are not abstractive sets, since the former do not have to satisfy (**??**).

Definition 4.1. If X, Y are subsets of a $\mathfrak{R} \in \mathsf{BPCA}$, then X is *coinitial* with Y ($X \trianglelefteq Y$) iff for every $y \in Y$ there is $x \in X$ such that $x \leq y$.

Definition 4.2. Abstractive sets A_1 and A_2 are *similar* ($A_1 \sim A_2$) iff $A_1 \trianglelefteq A_2$ and $A_2 \trianglelefteq A_1$.[2]

[1] Both the term and its abbrevation adopted from [**?**].

[2] Our co-initiality relation is Whitehead's covering relation. Thus A_1 and A_2 are similar iff A_1 covers A_2 and A_2 covers A_1.

Coinitiality of abstractive sets does not have to be equivalence relation, since it is not in general symmetric. However, it is reflexive and transitive, so similarity of abstractive sets must be an equivalence relation. After Whitehead, we will call every element of $\mathbf{A}/_{\sim}$ a *geometrical element*, and we will denote it by '$[A]$', for $A \in \mathbf{A}$. If $A_1, A_2 \in \mathbf{A}$, define:

$$[A_1] \preceq [A_2] \;:\longleftrightarrow\; A_1 \trianglelefteq A_2\,.$$

Definition 4.3. For $A \in \mathbf{A}$, $[A]$ is a *Whitehead point* (*W-point*) iff $[A]$ is minimal in $\langle \mathbf{A}/_{\sim}, \preceq\rangle$. The set of all Whitehead points will be denoted by '$\mathbf{W}$'.

$A \in \mathbf{A}$ is *W-representative* of a point iff $[A] \in \mathbf{W}$. Let $\mathbf{Q}_W$ be the set of all W-representatives of a given BCA.

5 Grzegorczyk vs. Whitehead

The first comparison between the two constructions was carried out in [?], and the purpose of this talk is to improve some results from the aforementioned paper. In particular we show:

Theorem 5.1. (**BPCA**) $\mathbf{Q}_G \cap \mathbf{A} \subseteq \mathbf{Q}_W$. *So, if there are no atoms and there are abstractive sets,* $\mathbf{Q}_G \subseteq \mathbf{Q}_W$.

With additional assumptions we can demonstrate that every W-point is a G-point as well. To this end, we will need the so-called *interpolation* axiom:

$$x \ll y \longrightarrow \exists_{z \in R}\, x \ll z \ll y\,, \tag{IA}$$

and the *connection* axiom:

$$x \notin \{0, 1\} \longrightarrow x \,\mathsf{C}\, {-x} \tag{C6}$$

Definition 5.1. For a given chain C let the *co-initiality* of C be the smallest cardinal number κ such that there exists an antitone function $f\colon \kappa \to C$ with $f[\kappa]$ co-initial with C.

Theorem 5.2. *If* $\mathfrak{B} \in \mathbf{BCA} + (\mathbf{??}) + (\mathbf{??})$, *then every Whitehead representative with co-initiality* ω *is a Grzegorczyk representative.*

We also prove that (**??**) is relevant for the theorem above by showing that:

Theorem 5.3. *There is* $\mathfrak{B} \in \mathbf{BCA} + (\mathbf{??})$ *with a countable W-representative, which is not a G-representative.*

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